ME 314 - Engineering Design : Mechanical Components
Lecture 5

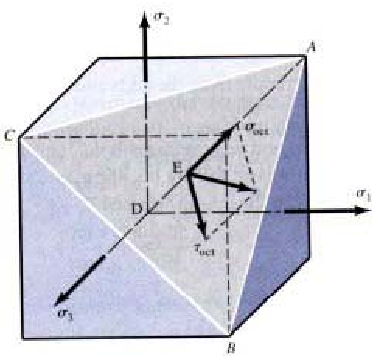
Note Title

Chapter 4 - Stress, Strain, and Deflection (Continued)

Octahedral Plane & Octahedral Stresses

A plane whose normal makes equal angles with the principal axes of stress is called the octahedral plane. The normal (σ_{oct}) and shear (τ_{oct}) stresses acting on this plane are as shown. As we shall see in Chapter 5, the von Mises stress that appears in failure theories for ductile materials is proportional to octahedral shear stress.

Octahedral Stresses



$$\sigma_{oct} = \frac{1}{3} I_1 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$\tau_{oct} = \frac{2}{3}\left(\tau_{1,2}^2 + \tau_{2,3}^2 + \tau_{1,3}^2\right)^{1/2}$$

$$= \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

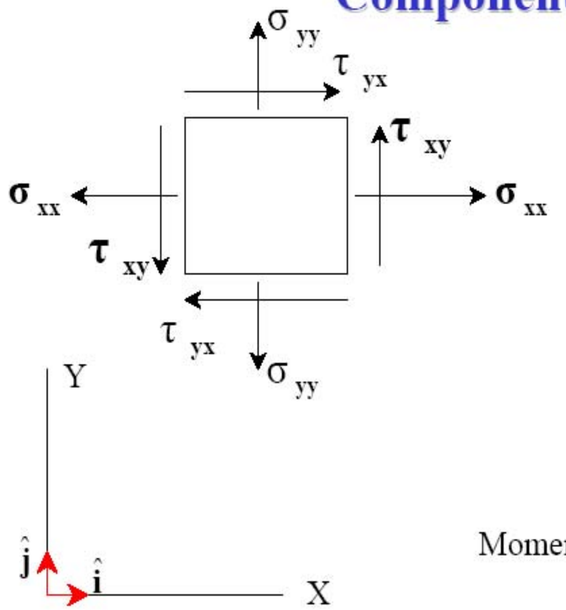
$$= \frac{1}{3}\begin{bmatrix} (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 \\ + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right) \end{bmatrix}^{1/2}$$

Note that there eight corner planes in a cube.
Hence the name octahedral stress.

Plane Stress

A very common state of stress occurs when the stresses on one surface are zero. This stress state is referred to as "plane stress" and can be shown on only one face of the cube.

2D Cartesian Stress Components



Notation

σ Normal Stress

τ Shear Stress

τ_{xy}

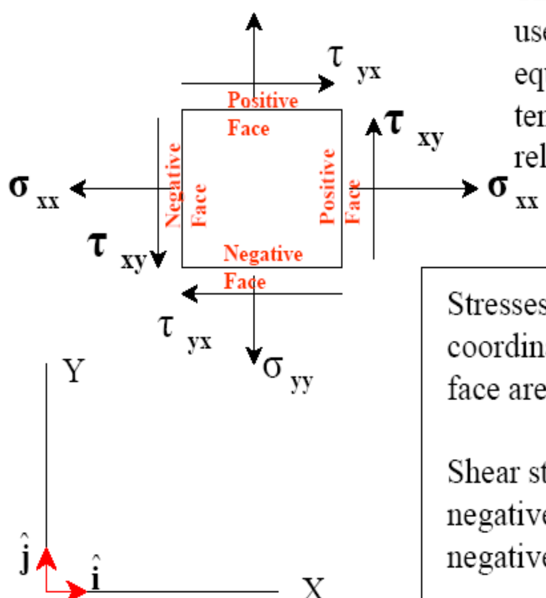
Face — Direction

Moment equilibrium requires that

$$\tau_{xy} = \tau_{yx}$$

Tensor Sign Convention

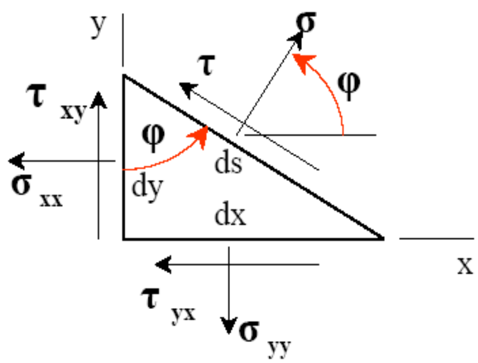
This sign convention must be used to satisfy the differential equilibrium equations and tensor transformation relationships.



Stresses acting in a positive coordinate direction on a positive face are positive.

Shear stresses acting in the negative coordinate direction on a negative face are positive.

2D Mohr's Circle
(Transformation of Axis)



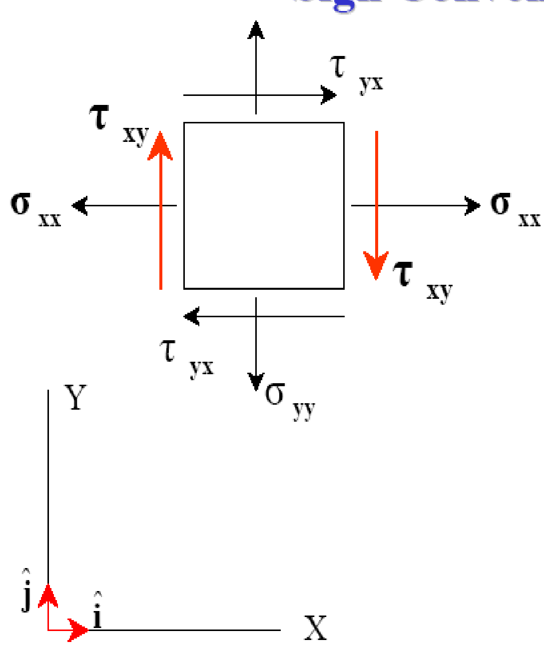
All equations for a 2-D Mohr's Circle are derived from this figure.

ΣF in the x- and y-directions yields the transformation-of-axis equations

$$\sigma = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos(2\varphi) + \tau_{xy}\sin(2\varphi)$$

$$\tau = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin(2\varphi) + \tau_{xy}\cos(2\varphi)$$

2D Mohr's Circle
Sign Convention



The sign convention used with the 2D Mohr's circle equations is slightly different.

A positive shear stress is one that tends to create clockwise (CW) rotation.

2D Mohr's Circle

(Principal Stress Equations)

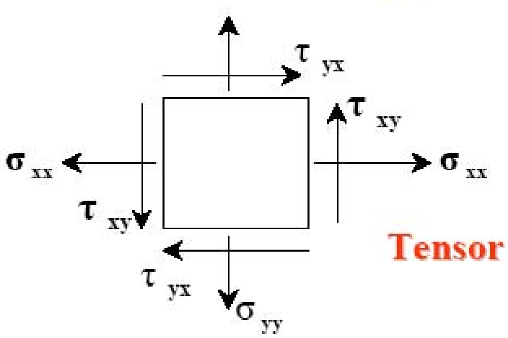
The transformation-of-axis equations can be used to find planes for which the normal and shear stress are the largest.

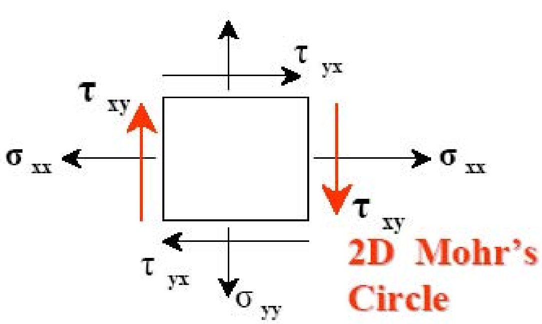
$$\sigma_1, \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_1, \tau_2 = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

We will use these equations extensively during this class.

Comments on Shear Stress Sign Convention





$$\sigma_1, \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_1, \tau_2 = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

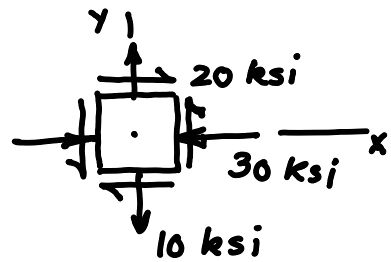
The sign convention is important when the transformation-of-axis equations are used.

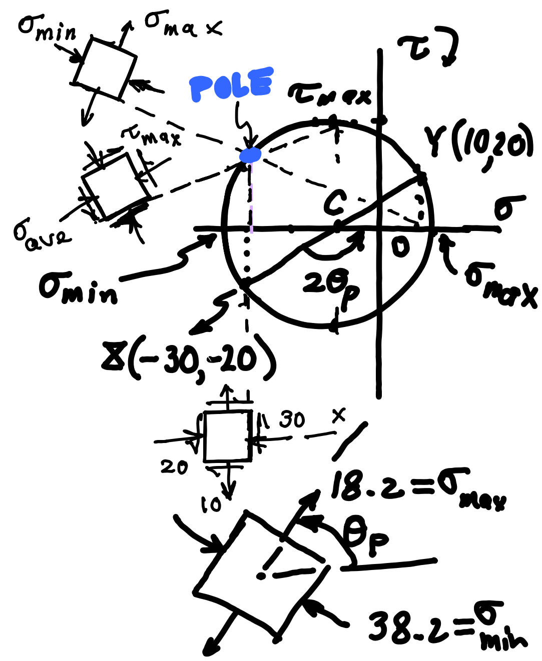
The same answer is obtained when computing the principal stress components.

Example: **Determine the principal stresses if the state of plane stress is given as:**

$\sigma_x = -30$ kpsi , $\sigma_y = 10$ kpsi ,

$\tau_{xy} = \tau_{yx} = 20$ kpsi





Analytically :

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{-30+10}{2} \pm \sqrt{\left(\frac{-30-10}{2}\right)^2 + 20^2} = -10 \pm \sqrt{800} = 18.2, -38.2 \text{ kpsi}$$

$$\tan 2\theta_p = \frac{2(20)}{-30-10} = -1 \Rightarrow \theta_p = 67.5^\circ, 157.5^\circ$$

4.2 Strain

In real life, there are no "rigid" bodies. However, if a body is sufficiently insensitive to deformation for the case considered, then we may make the decision to treat the body as rigid. Otherwise, deformation cannot be ignored. In fact, very often, strain and deflection rather than stress are the controlling factors in design.

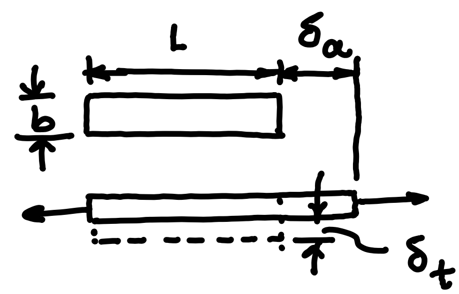
For example, in a transmission, gears are supported by a shaft. If the shaft bends too much (i.e., it is too "flexible") the gear teeth will not mesh properly. This can result in excessive wear, noise, and early failure.

An important property of strain and deflection is that they are directly measurable quantities.

Note that stress, on the other hand, cannot be directly measured. Stresses are calculated from measured strains by employing a stress-strain relation such as the Hook's law.

Normal Strain, ϵ, is the change in length per original length:

$$\epsilon = \frac{\text{change in length}}{\text{Original length}}$$



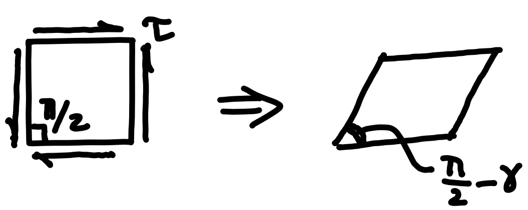
$$\epsilon_a = \frac{\delta_a}{L} = \text{Axial Strain}$$

$$\epsilon_t = -\frac{\delta_t}{b} = \text{Lateral Strain}$$

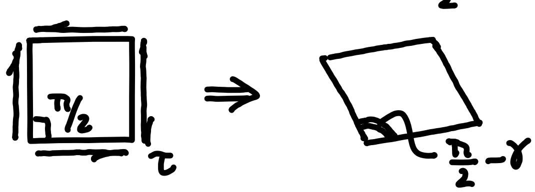
The Poisson's ratio is defined as

$$\nu = -\frac{\epsilon_t}{\epsilon_a}$$

Shear Strain, γ, is the change in a right angle of an element subjected to pure shear:



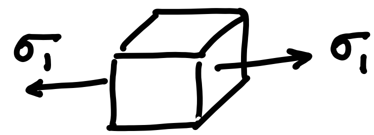
("Positive" Shear)



("Negative" Shear)

Stress-Strain & Strain-Stress Relations:

$\sigma_2 = \sigma_3 = 0$



Uniaxial Loading

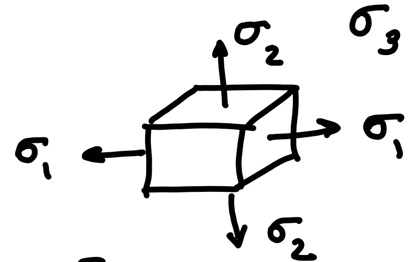
$$\epsilon_1 = \frac{\sigma_1}{E} \quad , \quad \epsilon_2 = \epsilon_3 = -\nu\,\epsilon_1 = -\nu\frac{\sigma_1}{E}$$

Invert:

$$\sigma_1 = E\epsilon_1 \quad , \quad \sigma_2 = \sigma_3 = 0$$

Biaxial Loading

$\sigma_3 = 0$



$$\epsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E}$$

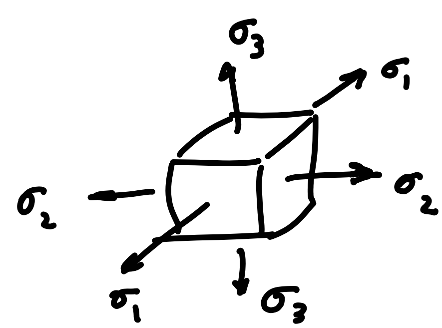
$$\epsilon_2 = \frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E} \quad , \quad \epsilon_3 = -\nu\frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E}$$

Invert:

$$\sigma_1 = \frac{E(\epsilon_1 + \nu\epsilon_2)}{1-\nu^2} \qquad \sigma_2 = \frac{E(\epsilon_2 + \nu\,\epsilon_1)}{1-\nu^2} \qquad \sigma_3 = 0$$

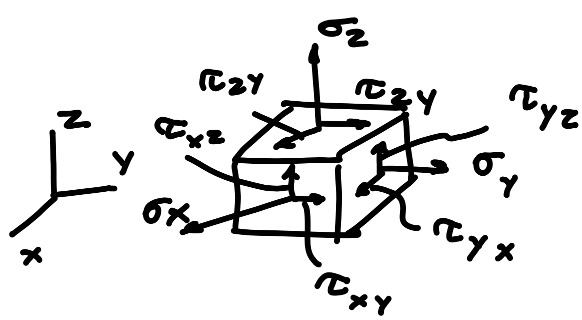
Triaxial Loading

$$\left.\begin{aligned}
\epsilon_1 &= \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} - \nu\frac{\sigma_3}{E} \\
\epsilon_2 &= \frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E} - \nu\frac{\sigma_3}{E} \\
\epsilon_3 &= \frac{\sigma_3}{E} - \nu\frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E}
\end{aligned}\right\}$$



Invert: $$\sigma_1 = \frac{E}{1-\nu-2\nu^2}\left[(1-\nu)\,\epsilon_1 + \nu(\epsilon_2 + \epsilon_3)\right] \text{, etc.}$$

General Loading

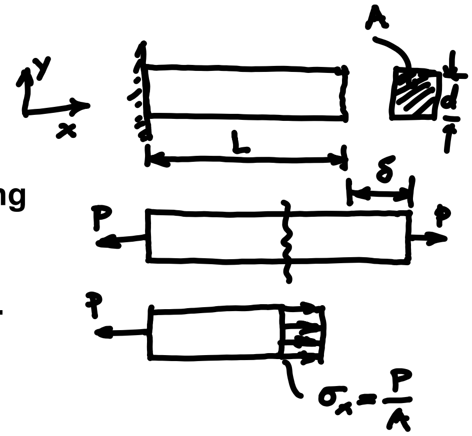


$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \text{, etc.}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \text{, etc.}$$

4.7 Axial Stress

These stresses could be either tensile or compressive and are caused by forces applied along the axis.



This stress is uniformly distributed only if

1) the section is far away from ends (at least 3d)
2) the load is applied along the axis otherwise bending moment develops
3) the bar is prismatic with no holes, notches, or imperfections that cause Stress concentration.
4) the material is homogeneous (not a composite)
5) the bar is free of residual stresses
6) the bar is in stable equilibrium (no buckling)

If the cross-sectional area is A, we have

$$\sigma_x = \frac{P}{A}$$

where P is the applied load at the centroid of the cross-section.

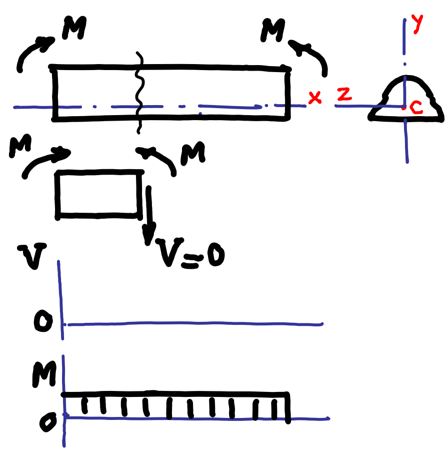
4.8 Direct Shear Stress, Bearing Stress, and Tearout

(Please study on your own)

4.9 Beams and Bending Stresses

Any element subjected to transverse loads will act as a beam. Cross section of a beam will be subjected to normal and shear stresses in the general case.

Pure Bending Assumptions:M



1. The section analyzed is not near applied loads or stress raisers.

2. Material is homogeneous, isotropic, linear elastic (i.e., obeys Hooke's law)
3. The beam is loaded in a plane of symmetry

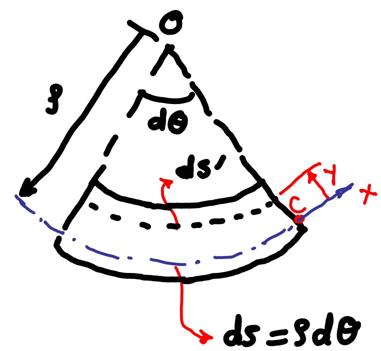
4. Plane sections remain plane & perpendicular to the neutral axis

5. deflections are small and less than the elastic limit

6. No axial or shear loads applied to segment

7. The beam is initially straight

Strain: $$\epsilon = \frac{ds' - ds}{ds} = \frac{(-y+\rho)\,d\theta - \rho\, d\theta}{\rho\, d\theta} = -\frac{y}{\rho}$$



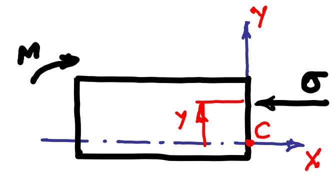
Stress-Strain: $$\sigma = E\epsilon = -\frac{Ey}{\rho}$$

Equilibrium:

$$\Sigma F = 0 \Rightarrow \int_A \sigma\, dA = 0$$

$$\Sigma M = 0 \Rightarrow M = -\int_A y\, \sigma\, dA$$

$$\therefore \int_A -\frac{Ey}{\rho}\, dA = 0 \Rightarrow \int_A y\, dA = 0,$$



i.e., **the neutral axis passes through the centroid of the cross-section**, and

$$M = -\int_A y\left(-\frac{Ey}{\rho}\right) dA = \frac{E}{\rho}\int_A y^2\, dA = \frac{EI}{\rho}$$

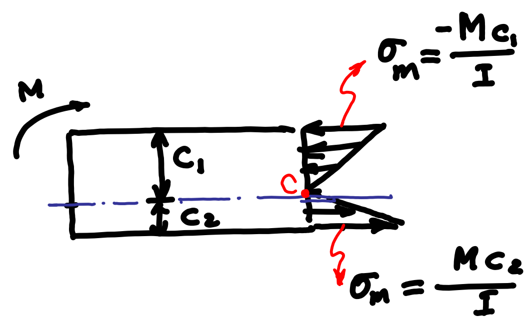
or
$$\frac{M}{EI} = \frac{1}{\rho} \qquad \text{(Beam Equation)}$$

Since $\sigma = -\frac{Ey}{\rho}$ or $\frac{1}{\rho} = -\frac{\sigma}{Ey}$, We have

$$\frac{M}{EI} = -\frac{\sigma}{Ey} \Longrightarrow \sigma = -\frac{My}{I}$$

Define the **Section Modulus**, **Z**, as

$$Z = \frac{I}{c}$$

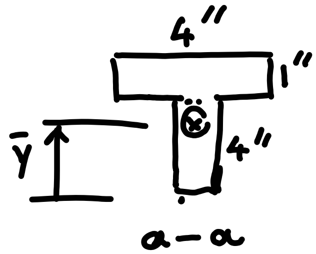


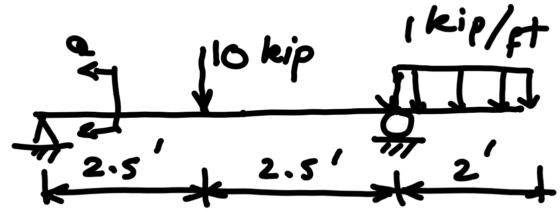
then

$$\sigma_{max} = \frac{M}{Z}$$

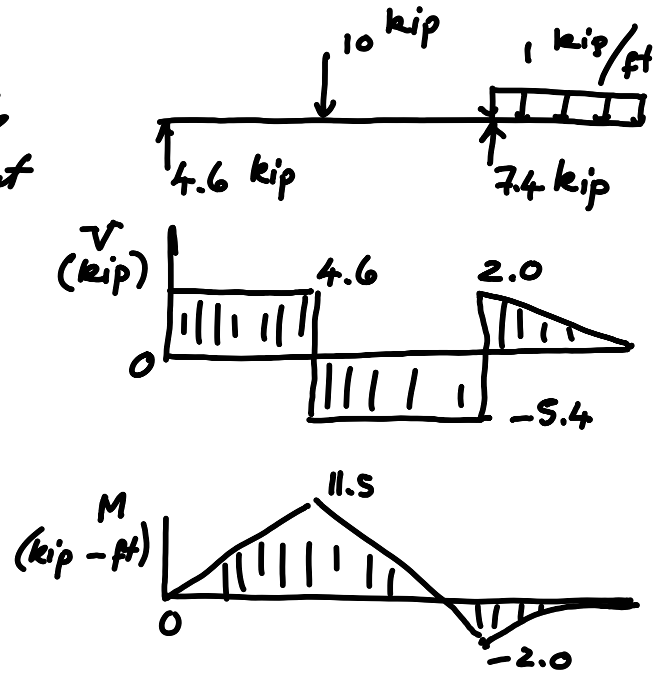
Geometrical properties A, I, and Z are given in Appendix A (?) and the CD that came with text (see page 157)

Example: Determine the maximum bending stress and its location for the beam & loading shown:





Solution: compute reactions & plot shear & moment diagrams



Cross Section Properties

Centroid:

$$\bar{y} = \frac{\Sigma yA}{\Sigma A} = \frac{4(1)(2) + 4(1)(4.5)}{(4)(1) + (4)(1)}$$

$$= 3.25 \text{ in}$$

Second Moment of Area:

$$I_{T} = I_{|} + I_{-} \quad , \quad I = I_0 + d^2 A$$

$$I_{|} = \frac{(1)(4)^3}{12} + (1.25)^2(4) = 11.6 \text{ in}^4$$

$$I_{-} = \frac{(4)(1)^3}{12} + (1.25)^2(4) = 6.6 \text{ in}^4$$

$$I_{T} = 11.6 + 6.6 = 18.2 \text{ in}^4$$

Max. Bending Stress

$$\sigma_{max} = \frac{Mc}{I}$$

There are no shear stresses developed under pure bending (since V = 0). Under transverse loading, however, shear stresses develop as shown in the following Example.